Math 250/Test 3/F' 13/100pts total
Burger

HP 0 95.

1. (20pts) Let $\alpha = \sqrt[3]{2} + \sqrt[3]{4}$. Prove that $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt[3]{2})$.

Let $\beta = \sqrt[3]{2}$. Then $\beta^3 - 2 = 0$. $x^3 - 2$ is irreducible by Eisenstein's Criterion, so $m_\beta(x) = x^3 - 2$.

20. Thus, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ (a basis for $\mathbb{Q}(\sqrt[3]{2})$ is $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$).

Now, $\alpha = 0 + 1 \cdot \sqrt[3]{2} + 1 \cdot \sqrt[3]{4} \in \mathbb{Q}(\sqrt[3]{2})$.

Therefore, $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt[3]{2})$. □ Nice!

2. (20pts) Find the multiplicative inverse of $1 + 5\sqrt{2}$ in $\mathbb{Q}(\sqrt{2})$.

We need $(1+5\sqrt{2})(a+b\sqrt{2}) = 1$ for some $a, b \in \mathbb{Q}$.

20. $(1+5\sqrt{2})(a+b\sqrt{2}) = 1 \Rightarrow (a+10b) + (5a+b)\sqrt{2} = 1 \Rightarrow a+10b = 1$ and $5a+b=0$

$\Rightarrow b = -5a \Rightarrow a + 10(-5a) = 1 \Rightarrow -49a = 1 \Rightarrow a = -\frac{1}{49}$

$\Rightarrow b = \frac{5}{49}$.

Thus, the multiplicative of $(1+5\sqrt{2})$ over $\mathbb{Q}(\sqrt{2})$ is $\boxed{-\frac{1}{49} + \frac{5}{49}\sqrt{2}}$.

Awesome!

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3. (15pts) True or False?

a. $\mathbb{C}$ is the splitting field for x^2+1 over $\mathbb{Q}$ T F

$\mathbb{Q}(i)$ is the splitting field for x^2+1 over $\mathbb{Q}$.

15 ✓

yes. The splitting field is the 'smallest' one containing the α's.

b. $[\frac{Q[x]}{\langle 14x^2+29x-15\rangle}:\mathbb{Q}]=2$ T F

The rational roots of $p(x):=14x^2+29x-15$ are of the form $\pm\frac{15,5,3,1}{14,7,2,1}$.
We find that $-\frac{5}{2}$ is a root. So $(x+\frac{5}{2})\mid p(x)$.

$$\begin{array}{c|ccc} -\frac{5}{2} & (14) & (29) & (-15) \\ & & (-35) & (15) \\ \hline & (14) & (-6) & (0) \end{array}$$

$\Longrightarrow p(x)=(x+\frac{5}{2})(14x-6)$.

Thus, $\mathbb{Q}[x]/\langle p(x)\rangle \cong \mathbb{Q}$.

Hence, $[\mathbb{Q}[x]/\langle p(x)\rangle : \mathbb{Q}] = [\mathbb{Q}:\mathbb{Q}]=1$

Also: $14x^2+29x-15=(7x-3)(2x+5)$ by ac-method.

~~c.~~ $[\frac{Q[x]}{\langle x^6+5x^3+2x^2+4x+5\rangle}:\mathbb{Q}]=6$ T F

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20 /25

4. (25pts)

a. Find the minimal polynomial of $x = \sqrt[3]{2} + \sqrt[3]{3}$ over $\mathbb{Q}$. (Hint: look for an x —substitution during your calculation)

$x = \sqrt[3]{2} + \sqrt[3]{3} \Rightarrow x^3 = 2 + \sqrt[3]{12} + 2\sqrt[3]{12} + 2\sqrt[3]{18} + \sqrt[3]{18} + \sqrt[3]{27} = 5 + 3\sqrt[3]{6}\underline{(\sqrt[3]{2}+\sqrt[3]{3})}$

x ·

$\Rightarrow x^3 - 5 = 3\sqrt[3]{6}\,x \Rightarrow x^9 - 15x^6 + 75x^3 - 125 = 162x^3$

+5

$\Rightarrow x^9 - 15x^6 - 87x^3 - 125 = 0.$

+10

If there are rational roots, they are one of : $\pm 1, 5, 25, 125$.
After trial-and-error (on the calculator), we see that there are no rational roots.
Thus, $\boxed{x^9 - 15x^6 - 87x^3 - 125}$ is irreducible, and hence, it is the minimal polynomial of $\sqrt[3]{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

−5

As we have been finding in class, RRT is not enough to determine irreducibility. As an example $(x^2+1)(x^2+2) = x^4 + 3x^2 + 2$ would fail RRT, but is reducible.

let $p = 3$; $3 \mid -15$ $3 \mid 87$ and $3^2 \nmid 125$
so By E-crit. it is irreducible.

b. Compute $[\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) : \mathbb{Q}(\sqrt[3]{2})]$ and draw a labeled field tower diagram with $\mathbb{Q}$ as the base field (see below).

+5.

From ⓐ, $[\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}) : \mathbb{Q}] = 9$. Since $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, and since $[\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}) : \mathbb{Q}(\sqrt[3]{2})][\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$, we get that $[\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}) : \mathbb{Q}(\sqrt[3]{2})] = \boxed{3}$.

$$\begin{array}{c} \mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{3}) \\ \Big| \; 3 \\ \mathbb{Q}(\sqrt[3]{2}) \\ \Big| \; 3 \\ \mathbb{Q} \end{array}$$

9.

✓

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5. (20pts) One ordered basis for the field $\mathbb{Q}(\sqrt[4]{3})$ is $\{1, \alpha, \alpha^2, \alpha^3\}$, where $\alpha = \sqrt[4]{3}$.

20/20

Another ordered basis for this degree 4 field extension over $\mathbb{Q}$ is:

$$\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{1, \quad 1-\alpha, \quad \alpha+\alpha^2, \quad 1+\alpha+\alpha^2-\alpha^3\}$$

It is known that $\frac{3}{2}\cdot\sqrt[4]{48} + \frac{2}{3}\sqrt[4]{729} + 6 + \sqrt[4]{27} \in \mathbb{Q}(\sqrt[4]{3})$. Write this element in the β-basis:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right].$$

Now, $\frac{3}{2}\cdot\sqrt[4]{48} + \frac{2}{3}\sqrt[4]{729} + 6 + \sqrt[4]{27} = \frac{3}{2}\cdot 2\cdot\sqrt[4]{3} + \frac{2}{3}\cdot 3\cdot\sqrt[4]{9} + 6 + \sqrt[4]{27}$

$= 6 + 3\sqrt[4]{3} + 2\sqrt[4]{9} + \sqrt[4]{27} = 6 + 3\alpha + 2\alpha^2 + \alpha^3.$

correct inverse!

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Therefore, $\frac{3}{2}\cdot\sqrt[4]{48} + \frac{2}{3}\sqrt[4]{729} + 6 + \sqrt[4]{27} = \boxed{8\beta_1 - \beta_2 + 3\beta_3 - \beta_4}$.

EXAM OVER